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Trigonometric Functions ⇆ Hyperbolic Functions

Construction of relationships that transform hyperbolic functions into trigonometric functions.

The Pythagorean formula for a right triangle with hypotenuse "h" and side "a" adjacent to angle α and side "b" opposite angle α is:

$$h^2 = a^2 + b^2 \qquad 33.20$$

For this triangle we have the following trigonometric functions $ft = ft(\alpha)$ with angle α:

$$a = h.cos\alpha \qquad b = h.sen\alpha \qquad 33.21$$

Reshaping the Pythagorean formula gives:

$$h^2 = a^2 + b^2 \to b^2 = h^2 - a^2 = (h+a)(h-a) \to \left(\frac{h+a}{b}\right)\left(\frac{h-a}{b}\right) = e^{\emptyset}e^{-\emptyset} = 1 \qquad 33.22$$

This is divided into the following hyperbolic functions $fh = fh(\emptyset)$ with angle $\emptyset$:

$$e^{\emptyset} = \frac{h+a}{b} > zero \qquad 33.23$$

$$e^{-\emptyset} = \frac{h-a}{b} > zero \qquad 33.24$$

Where applying the trigonometric functions we obtain $fh(\emptyset) = ft(\alpha)$:

$$e^{\emptyset} = \frac{h+a}{b} = \frac{h+h.cos\alpha}{h.sen\alpha} = \frac{1+cos\alpha}{sen\alpha} \qquad 33.25$$

$$e^{-\emptyset} = \frac{h-a}{b} = \frac{h-h.cos\alpha}{h.sen\alpha} = \frac{1-cos\alpha}{sen\alpha} \qquad 33.26$$

The real equality of the functions $fh(\emptyset) = ft(\alpha)$ only occurs if the angle of the hyperbolic function is equal to the angle of the trigonometric function, that is, if $fh(\emptyset) = ft(\emptyset)$ where both are hyperbolic functions or $fh(\alpha) = ft(\alpha)$ where both are trigonometric functions.

From trigonometry we have:

$$tg\left(\frac{\alpha}{2}\right) = \frac{1-cos\alpha}{sen\alpha} = \frac{sen\alpha}{1+cos\alpha} = \sqrt{\frac{1-cos\alpha}{1+cos\alpha}} \qquad 33.27$$

Applying 27 we obtain the fundamental function of the trigonometric angle α as a function of the hyperbolic angle $\emptyset$, $\alpha = \alpha(\emptyset)$:

$$e^{\emptyset} = \frac{1+cos\alpha}{sen\alpha} = \frac{1}{tg\left(\frac{\alpha}{2}\right)} = \frac{1}{\sqrt{\frac{1-cos\alpha}{1+cos\alpha}}} = \sqrt{\frac{1+cos\alpha}{1-cos\alpha}} \qquad 33.28$$

$$e^{-\emptyset} = \frac{1-cos\alpha}{sen\alpha} = tg\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-cos\alpha}{1+cos\alpha}} \qquad 33.29$$

$$\alpha = 2arctg\left(e^{-\emptyset}\right) \qquad 33.30$$

The substitution of the angle 30 $\alpha = \alpha(\emptyset)$ in the trigonometric functions transforms it into hyperbolic functions with the necessary restrictions of existence, in the following form:

$$fh(\emptyset) = ft(\alpha) = ft[\alpha(\emptyset)] = ft(\emptyset) \rightarrow fh(\emptyset) = ft(\emptyset) \qquad 33.31$$

From 28 and 29 we obtain the fundamental formulas of the hyperbolic angle $\emptyset$ as a function of the trigonometric angle α, $\emptyset = \emptyset(\alpha)$:

$$ln\left(e^{\emptyset}\right) = ln\left[\frac{1}{tg\left(\frac{\alpha}{2}\right)}\right] \rightarrow \emptyset = \emptyset(\alpha) = ln\left[\frac{1}{tg\left(\frac{\alpha}{2}\right)}\right] \qquad 33.32$$

$$ln\left(e^{-\emptyset}\right) = ln\left[tg\left(\frac{\alpha}{2}\right)\right] \rightarrow -\emptyset = ln\left[tg\left(\frac{\alpha}{2}\right)\right] \rightarrow \emptyset = \emptyset(\alpha) = ln\left[\frac{1}{tg\left(\frac{\alpha}{2}\right)}\right] \qquad 33.33$$

The functions (30) $\alpha = \alpha(\emptyset)$ and (32) $\emptyset = \emptyset(\alpha)$ are inverses of each other.

The substitution of angle 32 $\emptyset = \emptyset(\alpha)$ in the hyperbolic functions transforms it into trigonometric functions with the appropriate existence restrictions, in the following form:

$$ft(\alpha) = fh(\emptyset) = fh\,[\emptyset(\alpha)] = fh(\alpha) \rightarrow ft(\alpha) = fh(\alpha) \qquad 33.34$$

In the unitary hyperbola $\mathrm{x}^2 - y^2 = 1$ applying the functions $\mathrm{x} = \mathrm{ch}\emptyset$ and $\mathrm{y} = \mathrm{sh}\emptyset$ we get:

$$\mathrm{x}^2 - y^2 = \mathrm{ch}^2\emptyset - \mathrm{sh}^2\emptyset = (\mathrm{ch}\emptyset + \mathrm{sh}\emptyset)(\mathrm{ch}\emptyset - \mathrm{sh}\emptyset) = e^{\emptyset}.e^{-\emptyset} = 1 \qquad 33.35$$

Breaking it down into two functions yields the hyperbolic cosine "$\mathrm{ch}\emptyset$" and hyperbolic sine "$\mathrm{sh}\emptyset$" functions:

$$\mathrm{ch}\emptyset + \mathrm{sh}\emptyset = e^{\emptyset} \rightarrow x = \mathrm{ch}\emptyset = \frac{e^{\emptyset}+e^{-\emptyset}}{2} \qquad 33.36$$

$$\mathrm{ch}\emptyset - \mathrm{sh}\emptyset = e^{-\emptyset} \rightarrow y = \mathrm{sh}\emptyset = \frac{e^{\emptyset}-e^{-\emptyset}}{2} \qquad 33.37$$

In 36 and 37 we have the fundamental properties of the hyperbolic functions.

Applying to the hyperbolic cosine $\mathrm{ch}\emptyset$, the previous variables are obtained:

$$x = ch\emptyset = \frac{e^{\emptyset}+e^{-\emptyset}}{2} = \frac{1}{2}\left(\frac{h+a}{b} + \frac{h-a}{b}\right) = \frac{h}{b} = \frac{h}{h.sen\alpha} = \frac{1}{sen\alpha} = cosec\alpha \qquad 33.38$$

Applying to the hyperbolic sine $\mathrm{sh}\emptyset$, the previous variables are obtained:

$$y = sh\emptyset = \frac{e^{\emptyset}-e^{-\emptyset}}{2} = \frac{1}{2}\left(\frac{h+a}{b} - \frac{h-a}{b}\right) = \frac{a}{b} = \frac{h.cos\alpha}{h.sen\alpha} = \frac{cos\alpha}{sen\alpha} = cotg\alpha \qquad 33.39$$

Applying the hyperbolic cosine $x = \mathrm{ch}\emptyset = cosec\alpha$ and the hyperbolic sine $y = \mathrm{sh}\emptyset = cotg\alpha$ to the unitary hyperbola equation $x^2 - y^2 = 1$ we get:

$$x^2 - y^2 = ch^2\emptyset - sh^2\emptyset = cosec^2\alpha - cotg^2\alpha = 1 \qquad 33.40$$

Which is a result of trigonometry.

With the relations of the hyperbolic cosine $\mathrm{ch}\emptyset$ and the hyperbolic sine $\mathrm{sh}\emptyset$ we can define the other relations between the trigonometric functions and the hyperbolic functions:

$$tgh\emptyset = \frac{sh\emptyset}{ch\emptyset} = \frac{\frac{cos\alpha}{sen\alpha}}{\frac{1}{sen\alpha}} = cos\alpha \quad 33.41$$

$$cotgh\emptyset = \frac{ch\emptyset}{sh\emptyset} = \frac{\frac{1}{sen\alpha}}{\frac{cos\alpha}{sen\alpha}} = \frac{1}{cos\alpha} = sec\alpha \quad 33.42$$

$$sech\emptyset = \frac{1}{ch\emptyset} = \frac{1}{\frac{1}{sen\alpha}} = sen\alpha \quad 33.43$$

$$cosech\emptyset = \frac{1}{sh\emptyset} = \frac{1}{\frac{cos\alpha}{sen\alpha}} = \frac{sen\alpha}{cos\alpha} = tg\alpha \quad 33.44$$

$$sech^2\emptyset + tgh^2\emptyset = sen^2\alpha + cos^2\alpha = 1 \quad 33.45$$

$$cotgh^2\emptyset - cosech^2\emptyset = sec^2\alpha - tg^2\alpha = 1 \quad 33.46$$

Construction of the already known relationships that transform the hyperbolic functions into the exponential form of a complex number.

Next, we will use Euler's formulas:

$$e^{i\alpha} = cos\alpha + isen\alpha \qquad e^{-i\alpha} = cos\alpha - isen\alpha \quad 33.47$$

Reshaping the Pythagorean formula, we get:

$$h^2 = a^2 + b^2 = a^2 - (ib)^2 = (a+ib)(a-ib) \rightarrow \frac{(a+ib)}{h}\frac{(a-ib)}{h} = e^{\emptyset}e^{-\emptyset} = 1 \quad 33.48$$

This breaks down into the following complex hyperbolic functions:

$$e^{\emptyset} = \frac{a+ib}{h} > zero \quad 33.49$$

$$e^{-\emptyset} = \frac{a-ib}{h} > zero \quad 33.50$$

For this triangle we have the trigonometric relations:

$$\frac{a}{h} = cos\alpha \qquad \frac{b}{h} = sen\alpha \quad 33.51$$

Applying trigonometric relations, we get:

$$e^{\emptyset} = \frac{a+ib}{h} = \frac{a}{h} + i\frac{b}{h} = cos\alpha + isen\alpha \quad 33.52$$

$$e^{-\emptyset} = \frac{a-ib}{h} = \frac{a}{h} - i\frac{b}{h} = cos\alpha - isen\alpha \quad 33.53$$

To conform to Euler's formulas we must change the hyperbolic arguments to $\emptyset = i\alpha$ and thus we obtain the hyperbolic functions written as the exponential form of a complex number:

$$e^{\emptyset} = e^{i\alpha} = cos\alpha + isen\alpha \quad 33.54$$

$$e^{-\emptyset} = e^{-i\alpha} = cos\alpha - isen\alpha \quad 33.55$$

Calling the cosseno $chi\alpha$ hyperbolic complex as:

$$x = chi\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2} = \frac{1}{2}[(cos\alpha + isen\alpha) + (cos\alpha - isen\alpha)] = cos\alpha \qquad 33.56$$

And naming the sine $shi\alpha$ hyperbolic complex as:

$$y = shi\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2} = \frac{1}{2}[(cos\alpha + isen\alpha) - (cos\alpha - isen\alpha)] = isen\alpha \qquad 33.57$$

Applying the cosine $x = chi\alpha = cos\alpha$ hyperbolic complex and the sine $y = shi\alpha = isen\alpha$ hyperbolic complex in the equation of the unit hyperbola $x^2 - y^2 = 1$ results:

$$x^2 - y^2 = ch^2 i\alpha - sh^2 i\alpha = cos^2\alpha - i^2 sen^2\alpha = cos^2\alpha + sen^2\alpha = 1 \qquad 33.58$$

Which is a result of trigonometry.

With the relationships of the hyperbolic cosine $chi\alpha = cos\alpha$ and the hyperbolic sine $shi\alpha = isen\alpha$ we can define the other relationships between complex trigonometric functions and complex hyperbolic functions.

Construction of relationships that transform hyperbolic functions into trigonometric functions similar to those that occur in Gudermannian functions.

The Pythagorean formula for a right triangle with hypotenuse “h” and side “a” adjacent to angle α and side “b” opposite angle α is:

$$h^2 = a^2 + b^2 \qquad 33.59$$

For this triangle we have the trigonometric relations:

$$a = h.cos\alpha \qquad\qquad b = h.sen\alpha \qquad 33.60$$

Reshaping the Pythagorean formula gives:

$$h^2 = a^2 + b^2 \rightarrow a^2 = h^2 - b^2 = (h+b)(h-b) \rightarrow \left(\frac{h+b}{a}\right)\left(\frac{h-b}{a}\right) = e^{\beta}.e^{-\beta} = 1 \qquad 33.61$$

This is divided into the following hyperbolic functions:

$$e^{\beta} = \frac{h+b}{a} > zero \qquad 33.62$$

$$e^{-\beta} = \frac{h-b}{a} > zero \qquad 33.63$$

Where applying the trigonometric relations we obtain:

$$e^{\beta} = \frac{h+b}{a} = \frac{h+h.sen\alpha}{h.cos\alpha} = \frac{1+sen\alpha}{cos\alpha} \qquad 33.64$$

$$e^{-\beta} = \frac{h-b}{a} = \frac{h-hsen\alpha}{h.cos\alpha} = \frac{1-sen\alpha}{cos\alpha} \qquad 33.65$$

From these we obtain the fundamental formulas of the hyperbolic angle β:

$$ln(e^{\beta}) = ln\left(\frac{1+sen\alpha}{cos\alpha}\right) \rightarrow \beta = ln\left(\frac{1+sen\alpha}{cos\alpha}\right) \qquad 33.66$$

$$ln\left(e^{-\beta}\right) = ln\left(\frac{1-sen\alpha}{cos\alpha}\right) \rightarrow \beta = -ln\left(\frac{1-sen\alpha}{cos\alpha}\right) \quad 33.67$$

Denominating the hyperbolic cosine $ch\beta$ as:

$$x = ch\beta = \frac{e^{\beta}+e^{-\beta}}{2} = \frac{1}{2}\left(\frac{h+b}{a} + \frac{h-b}{a}\right) = \frac{h}{a} = \frac{h}{h.cos\alpha} = \frac{1}{cos\alpha} = sec\alpha \quad 33.68$$

And calling the hyperbolic sine $sh\beta$ as:

$$y = sh\beta = \frac{e^{\beta}-e^{-\beta}}{2} = \frac{1}{2}\left(\frac{h+b}{a} - \frac{h-b}{a}\right) = \frac{b}{a} = \frac{h.sen\alpha}{h.cos\alpha} = \frac{sen\alpha}{cos\alpha} = tg\alpha \quad 33.69$$

Applying the hyperbolic cosine $x = ch\beta = sec\alpha$ and the hyperbolic sine $y = sh\beta = tg\alpha$ to the unitary hyperbola equation $x^2 - y^2 = 1$ we get:

$$x^2 - y^2 = ch^2\beta - sh^2\beta = sec^2\alpha - tg^2\alpha = 1 \quad 33.70$$

Which is a result of trigonometry.

With the relations of the hyperbolic cosine $ch\beta$ and the hyperbolic sine $sh\beta$ we can define the other relations between the trigonometric functions and the hyperbolic functions:

$$tgh\beta = \frac{sh\beta}{ch\beta} = \frac{\frac{sen\alpha}{cos\alpha}}{\frac{1}{cos\alpha}} = sen\alpha \quad 33.71$$

$$cotgh\beta = \frac{ch\beta}{sh\beta} = \frac{\frac{1}{cos\alpha}}{\frac{sen\alpha}{cos\alpha}} = \frac{1}{sen\alpha} = cosec\alpha \quad 33.72$$

$$sech\beta = \frac{1}{ch\beta} = \frac{1}{\frac{1}{cos\alpha}} = cos\alpha \quad 33.73$$

$$cosech\beta = \frac{1}{sh\beta} = \frac{1}{\frac{sen\alpha}{cos\alpha}} = \frac{cos\alpha}{sen\alpha} = cotg\alpha \quad 33.74$$

$$sech^2\beta + tgh^2\beta = cos^2\alpha + sen^2\alpha = 1 \quad 33.75$$

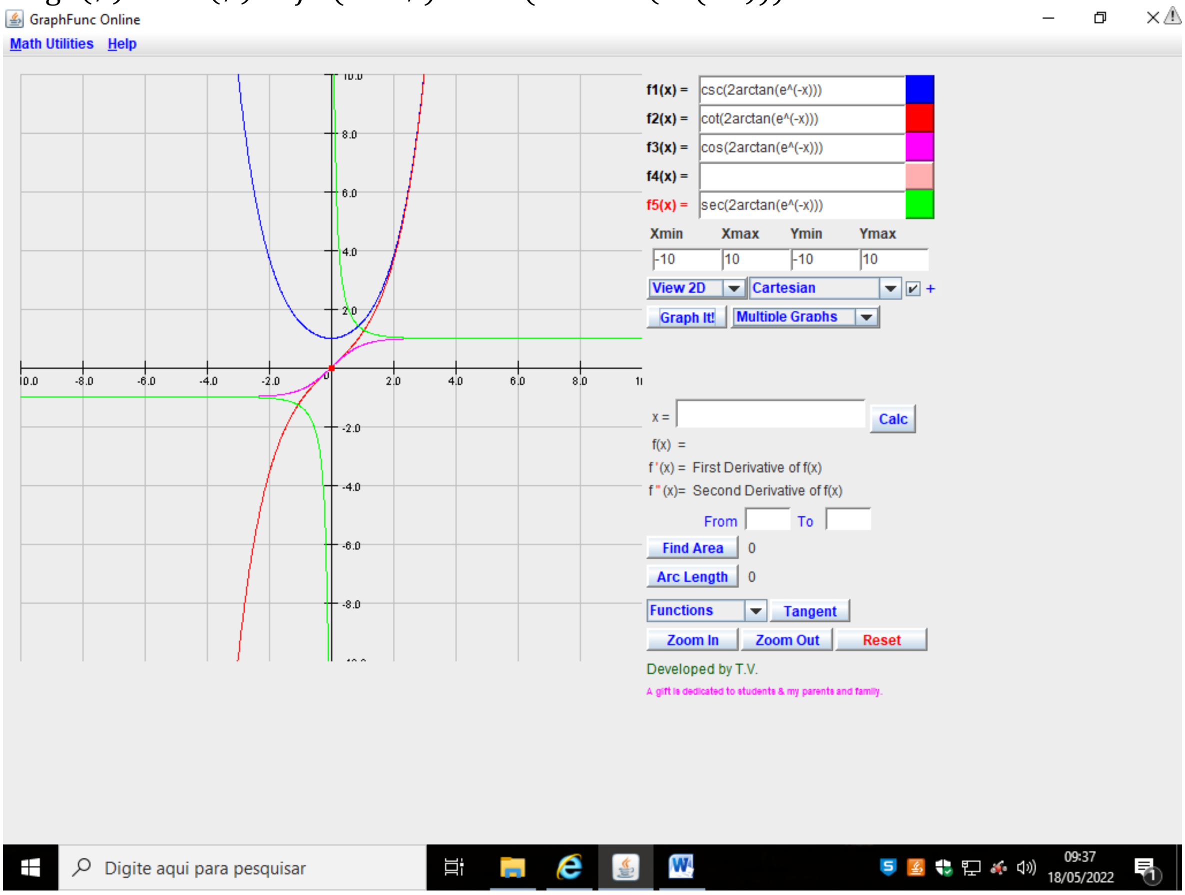
$$cotgh^2\beta - cosech^2\beta = cosec^2\alpha - cotg^2\alpha = 1 \quad 33.76$$

$\mathrm{ch}(\emptyset) = \csc(\emptyset) = f1(x = \emptyset) = \csc(2\arctan(e\wedge(-x)))$ 33.38

$\mathrm{sh}(\emptyset) = \cot(\emptyset) = f2(x = \emptyset) = \cot(2\arctan(e\wedge(-x)))$ 33.39

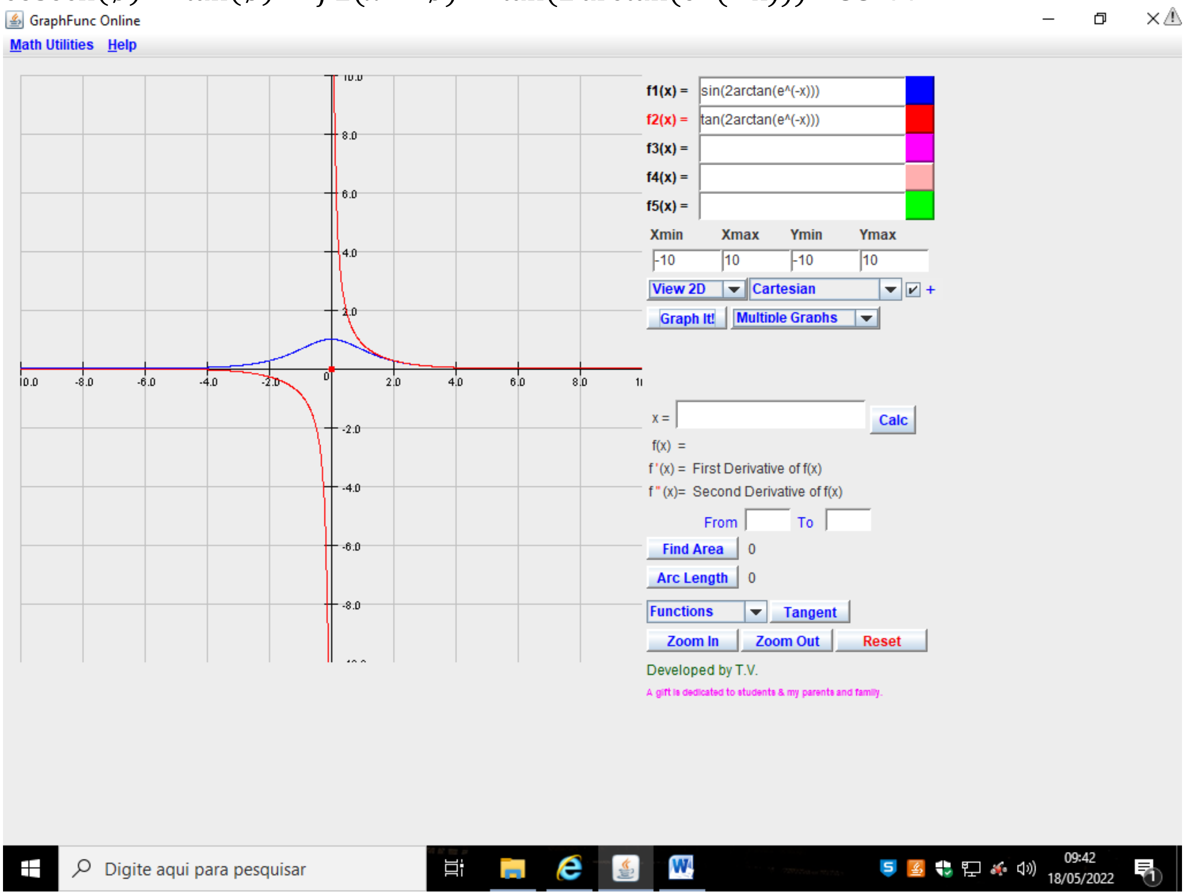
$\mathrm{tgh}(\emptyset) = \cos(\emptyset) = f3(x = \emptyset) = \cos(2\arctan(e\wedge(-x)))$ 33.41

$\mathrm{cotgh}(\emptyset) = \sec(\emptyset) = f5(x = \emptyset) = \sec(2\arctan(e\wedge(-x)))$ 33.42



$\mathrm{sech}(\emptyset) = \sin(\emptyset) = f1(x = \emptyset) = \sin(2\arctan(e\wedge(-x)))$ 33.43

$\mathrm{cosech}(\emptyset) = \tan(\emptyset) = f2(x = \emptyset) = \tan(2\arctan(e\wedge(-x)))$ 33.44

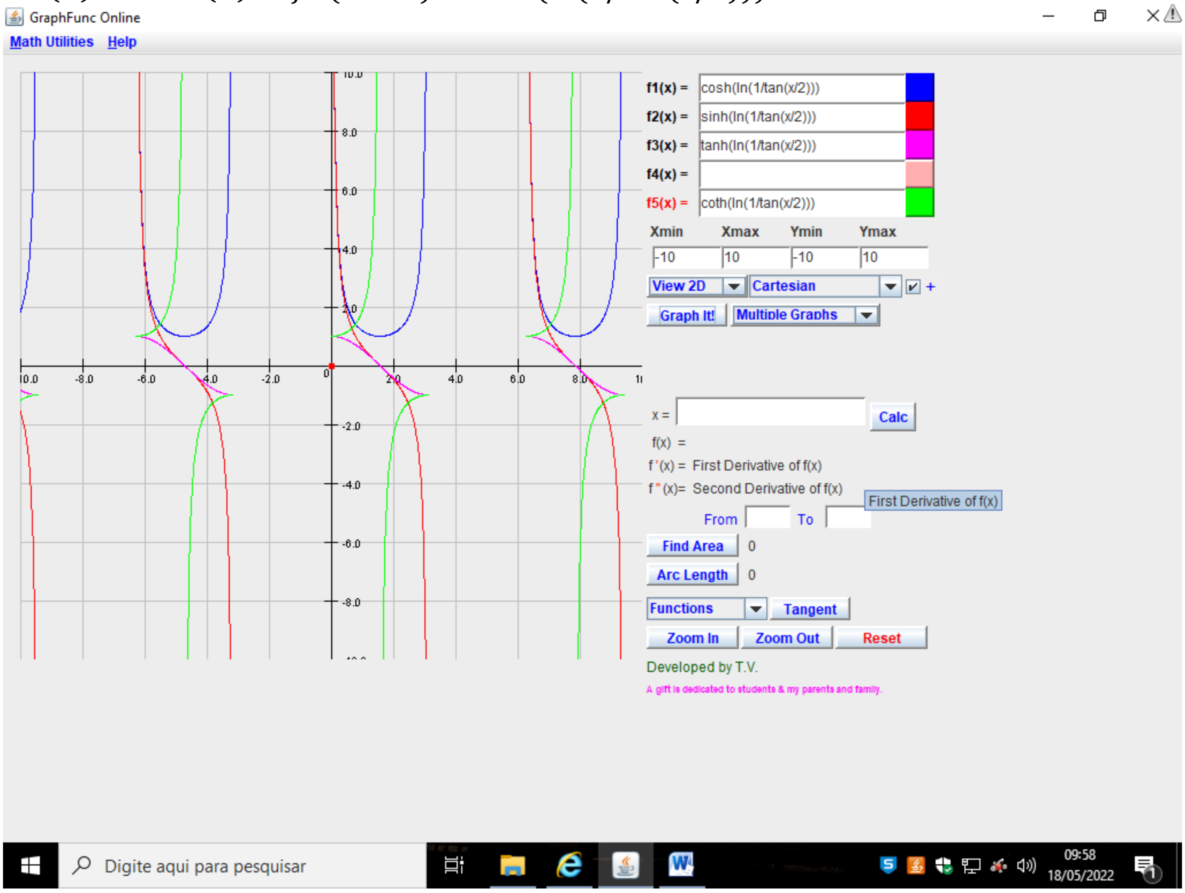


$$\operatorname{cosec}(\alpha) = \cosh(\alpha) = f1(x = \alpha) = \cosh(\ln(1/\tan(x/2))) \qquad 33.38$$

$$\operatorname{cotg}(\alpha) = \sinh(\alpha) = \mathrm{f2}(\mathrm{x} = \alpha) = \sinh(\ln(1/\tan(x/2))) \qquad 33.39$$

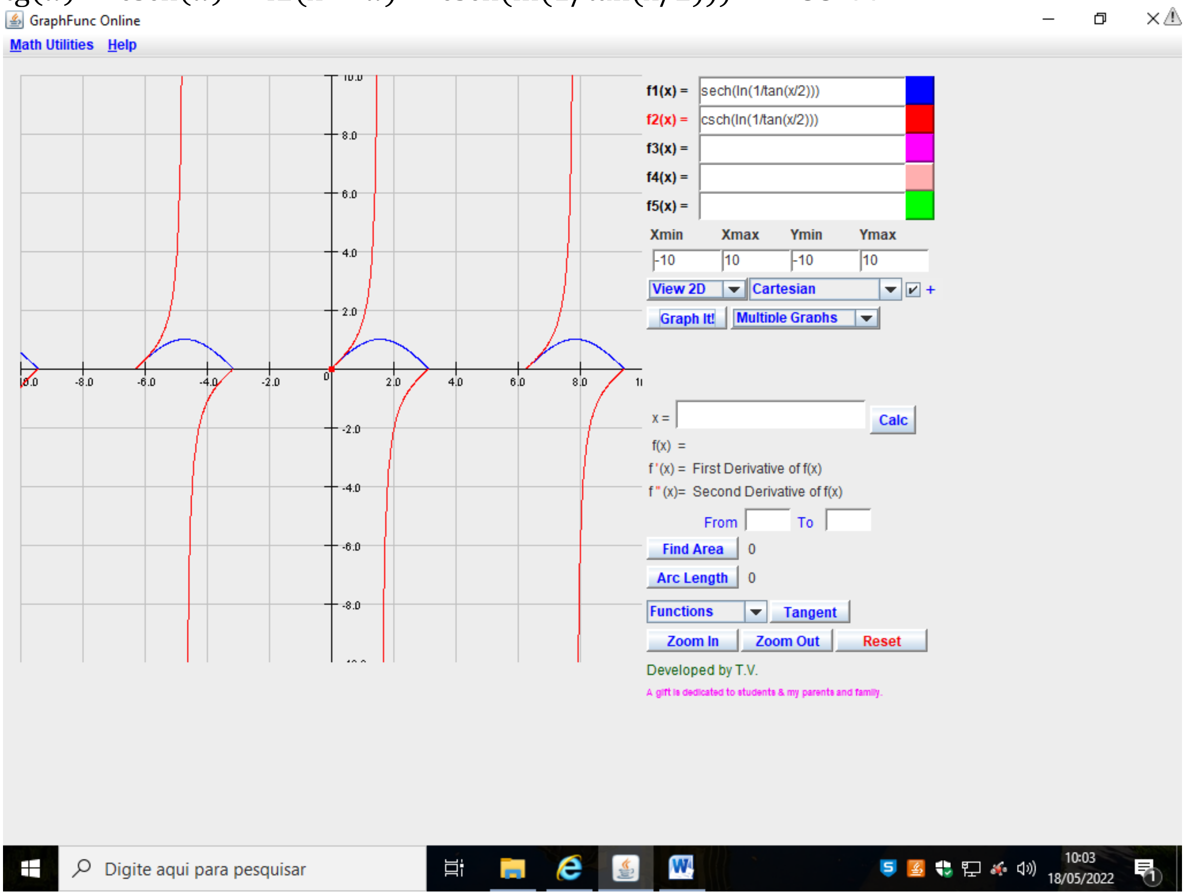
$$\cos(\alpha) = tanh\,(\alpha) = f3(x = \alpha) = \tanh(\ln(1/\tan(x/2))) \qquad 33.41$$

$$\sec(\alpha) = coth\,(\alpha) = f5(x = \alpha) = \coth(\ln(1/\tan(x/2))) \qquad 33.42$$



$$\sin(\alpha) = sech(\alpha) = f1(x = \alpha) = \operatorname{sech}(\ln(1/\tan(x/2))) \qquad 33.43$$

$$\operatorname{tg}(\alpha) = \operatorname{csch}(\alpha) = \mathrm{f2}(\mathrm{x} = \alpha) = \operatorname{csch}(\ln(1/\tan(x/2))) \qquad 33.44$$

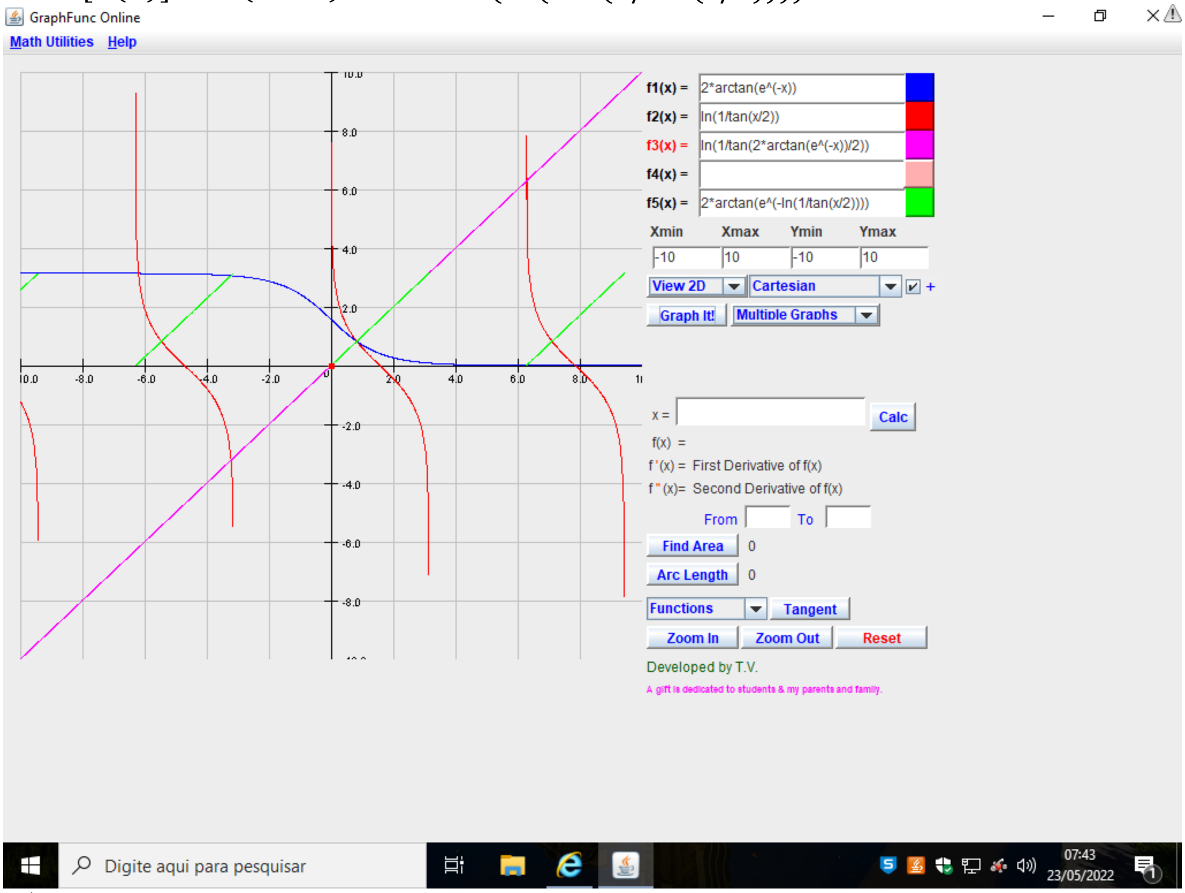


$\alpha = \alpha(\emptyset) = \mathrm{f1}(x = \emptyset) = \mathrm{2arctan(e\hat{}(-x))}$ 33.30

$\emptyset = \emptyset(\alpha) = f2(x = \alpha) = \mathrm{ln(1/tan(x/2))}$ 33.32

$\emptyset = \emptyset[\alpha(\emptyset)] = f3(x = \emptyset) = \mathrm{ln(1/tan(2arctan(e\hat{}(-x))/2))}$

$\alpha = \alpha[\emptyset(\alpha)] = \mathrm{f5(x = \alpha) = 2arctan(e\hat{}(-ln(1/tan(x/2))))}$



$e^{\emptyset} = f1(x = \emptyset) = \csc(2\arctan(\mathrm{e\hat{}(-x)})) + \cot(2\arctan(\mathrm{e\hat{}(-x)}))$ 33.36

$e^{-\emptyset} = f2(x = \emptyset) = \csc(2\arctan(\mathrm{e\hat{}(-x)})) - \cot(2\arctan(\mathrm{e\hat{}(-x)}))$ 33.37

$ch(\emptyset) = f3(x = \emptyset) = \csc(2\arctan(\mathrm{e\hat{}(-x)}))$ 33.38

$sh(\emptyset) = f5(x = \emptyset) = \cot(2\arctan(\mathrm{e\hat{}(-x)}))$ 33.39